

On the Dimension of the Stability Group for a Levi Non-Degenerate Hypersurface ^{*} [†]

V. V. Ezhov and A. V. Isaev

We classify locally defined Levi non-degenerate non-spherical real-analytic hypersurfaces in complex space for which the dimension of the group of local CR-automorphisms has the second largest positive value.

1 Introduction

Let M be a real-analytic hypersurface in \mathbb{C}^{n+1} passing through the origin. Assume that the Levi form of M at 0 is non-degenerate and has signature $(n - m, m)$ with $n \geq 2m$. Then in some local holomorphic coordinates $z = (z_1, \dots, z_n)$, $w = u + iv$ in a neighborhood of the origin, M can be written in the Chern-Moser normal form (see [CM]), that is, given by an equation

$$v = \langle z, z \rangle + \sum_{k, \bar{l} \geq 2} F_{k\bar{l}}(z, \bar{z}, u),$$

where $\langle z, z \rangle = \sum_{\alpha, \beta=1}^n h_{\alpha\beta} z_{\alpha} \bar{z}_{\beta}$ is a non-degenerate Hermitian form with signature $(n - m, m)$, and $F_{k\bar{l}}(z, \bar{z}, u)$ are polynomials of degree k in z and \bar{l} in \bar{z} whose coefficients are analytic functions of u such that the following conditions hold

$$\begin{aligned} \operatorname{tr} F_{2\bar{2}} &\equiv 0, \\ \operatorname{tr}^2 F_{2\bar{3}} &\equiv 0, \\ \operatorname{tr}^3 F_{3\bar{3}} &\equiv 0. \end{aligned} \tag{1.1}$$

Here the operator tr is defined as

$$\operatorname{tr} := \sum_{\alpha, \beta=1}^n \hat{h}_{\alpha\beta} \frac{\partial^2}{\partial z_{\alpha} \partial \bar{z}_{\beta}},$$

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where $(\hat{h}_{\alpha\beta})$ is the matrix inverse to $H := (h_{\alpha\beta})$. Everywhere below we assume that M is given in the normal form.

Let $\text{Aut}_0(M)$ denote the group of all local CR-automorphisms of M defined near 0 and preserving 0. To avoid confusion with the term “isotropy group of M at 0” usually reserved for global CR-automorphisms of M preserving the origin, this group is often called the *stability group* of M at 0. Every element φ of $\text{Aut}_0(M)$ extends to a biholomorphic mapping defined in a neighborhood of the origin in \mathbb{C}^{n+1} and therefore can be written as

$$\begin{aligned} z &\mapsto f_\varphi(z, w), \\ w &\mapsto g_\varphi(z, w), \end{aligned}$$

where f_φ and g_φ are holomorphic. We equip $\text{Aut}_0(M)$ with the topology of uniform convergence of the partial derivatives of all orders of the component functions on a neighborhood of 0. The group $\text{Aut}_0(M)$ with this topology is a topological group.

It follows from [CM] that every element $\varphi = (f_\varphi, g_\varphi)$ of $\text{Aut}_0(M)$ is uniquely determined by a set of parameters $(U_\varphi, a_\varphi, \lambda_\varphi, \sigma_\varphi, r_\varphi)$, where $\sigma_\varphi = \pm 1$, U_φ is an $n \times n$ -matrix such that $\langle U_\varphi z, U_\varphi z \rangle = \sigma_\varphi \langle z, z \rangle$ for all $z \in \mathbb{C}^n$, $a_\varphi \in \mathbb{C}^n$, $\lambda_\varphi > 0$, $r_\varphi \in \mathbb{R}$ (note that σ_φ can be equal to -1 only for $n = 2m$). These parameters are determined by the following relations

$$\begin{aligned} \frac{\partial f_\varphi}{\partial z}(0) &= \lambda_\varphi U_\varphi, \quad \frac{\partial f_\varphi}{\partial w}(0) = \lambda_\varphi U_\varphi a_\varphi, \\ \frac{\partial g_\varphi}{\partial w}(0) &= \sigma_\varphi \lambda_\varphi^2, \quad \text{Re} \frac{\partial^2 g_\varphi}{\partial^2 w}(0) = 2\sigma_\varphi \lambda_\varphi^2 r_\varphi. \end{aligned}$$

For results on the dependence of local CR-mappings on their jets in more general settings see [BER1], [BER2], [Eb], [Z].

We assume that M is *non-spherical at the origin*, i.e., that M in a neighborhood of the origin is not CR-equivalent to an open subset of the hyperquadric given by the equation $v = \langle z, z \rangle$. In this case for every element $\varphi = (f_\varphi, g_\varphi)$ of $\text{Aut}_0(M)$ the parameters $a_\varphi, \lambda_\varphi, \sigma_\varphi, r_\varphi$ are uniquely determined by the matrix U_φ , and the mapping

$$\Phi : \text{Aut}_0(M) \rightarrow GL_n(\mathbb{C}), \quad \Phi : \varphi \mapsto U_\varphi$$

is a topological group isomorphism between $\text{Aut}_0(M)$ and $G_0 := \Phi(\text{Aut}_0(M))$ with G_0 being a real algebraic subgroup of $GL_n(\mathbb{C})$; in addition the mapping

$$\Lambda : G_0(M) \rightarrow \mathbb{R}_+, \quad \Lambda : U_\varphi \mapsto \lambda_\varphi \tag{1.2}$$

is a Lie group homomorphism with the property $\Lambda(U_\varphi) = 1$ if all eigenvalues of U_φ are unimodular, where \mathbb{R}_+ is the group of positive real numbers with respect to multiplication (see [CM], [B], [L1], [BV], [VK]). Since $G_0(M)$ is a closed subgroup of $GL_n(\mathbb{C})$, we can pull back its Lie group structure to $\text{Aut}_0(M)$ by means of Φ (note that the pulled back topology is identical to that of $\text{Aut}_0(M)$). Let $d_0(M)$ denote the dimension of $\text{Aut}_0(M)$. We are interested in characterizing hypersurfaces for which $d_0(M)$ is large.

If $n > 2m$, $G_0(M)$ is a closed subgroup of the pseudounitary group $U(n - m, m)$ of all matrices U such that

$$U^t H \bar{U} = H,$$

(recall that H is the matrix of the Hermitian form $\langle z, z \rangle$). The group $U(n, 0)$ is the unitary group $U(n)$. If $n = 2m$, G_0 is a closed subgroup of the group $U'(m, m)$ of all matrices U such that

$$U^t H \bar{U} = \pm H,$$

that has two connected components. In particular, we always have $d_0(M) \leq n^2$. If $d_0(M) = n^2$ and $n > 2m$, then $G_0(M) = U(n - m, m)$. If $d_0(M) = n^2$ and $n = 2m$, then we have either $G_0(M) = U(m, m)$, or $G_0(M) = U'(m, m)$.

We will say that the group $\text{Aut}_0(M)$ is *linearizable*, if in some coordinates every $\varphi \in \text{Aut}_0(M)$ can be written in the form

$$\begin{aligned} z &\mapsto \lambda U z, \\ w &\mapsto \sigma \lambda^2 w. \end{aligned} \tag{1.3}$$

Clearly, in the above formula $U = U_\varphi$, $\lambda = \lambda_\varphi$, $\sigma = \sigma_\varphi$. The group $\text{Aut}_0(M)$ is known to be linearizable, for example, for $m = 0$ (see [KL]) and for $m = 1$ (see [Ezh1], [Ezh2]). If all elements of $\text{Aut}_0(M)$ in some coordinates have the form (1.3), we say that $\text{Aut}_0(M)$ is *linear* in these coordinates. It is shown in Lemma 3 of [Ezh3] that if $\text{Aut}_0(M)$ is linear in some coordinates, it is linear in some normal coordinates as well.

We will first discuss the case when $d_0(M)$ takes the largest possible value, that is, when $d_0(M) = n^2$. Observe that in this case $\text{Aut}_0(M)$ is linearizable for any m . Indeed, if $d_0(M) = n^2$, the group $G_0(M)$ contains $U(n - m, m)$. Hence $G_0(M)$ contains the subgroup $Q := \{e^{it} \cdot E_n, t \in \mathbb{R}\}$, where E_n is the $n \times n$ identity matrix. Let $\hat{Q} = \Phi^{-1}(Q) \subset \text{Aut}_0(M)$. The subgroup \hat{Q} is compact, and the argument in [KL] (see also [VK]) yields that in some normal coordinates every element of \hat{Q} can be written in the form (1.3). For every $\varphi \in \hat{Q}$ we clearly have $\sigma_\varphi = 1$. Further, since Q is

compact, there are no non-trivial homomorphisms from Q into \mathbb{R}_+ , and therefore $\lambda_\varphi = 1$ for every $\varphi \in \hat{Q}$. Hence, in these coordinates the function

$$F(z, \bar{z}, u) := \sum_{k, \bar{l} \geq 2} F_{k\bar{l}}(z, \bar{z}, u)$$

is invariant under all linear transformations from Q and thus $F_{k\bar{l}} \equiv 0$, if $k \neq \bar{l}$. We will now show that $\text{Aut}_0(M)$ is linearizable. Since linearizability arguments of this kind will occur several times throughout the paper, we give some details on the linearizability of $\text{Aut}_0(M)$ for general hypersurfaces.

Suppose that M is given in the Chern-Moser normal form near the origin. The main step in showing that $\text{Aut}_0(M)$ is linearizable is to prove that in some normal coordinates for every $\varphi \in \text{Aut}_0(M)$, we have $a_\varphi = 0$. Indeed, if $a_\varphi = 0$, it follows from [CM] that φ in the given coordinates is a fractional linear transformation that becomes linear if $r_\varphi = 0$. It is shown in the proof of Proposition 3 of [L2] that $a_\varphi = 0$ implies that $r_\varphi = 0$, provided $\lambda_\varphi = 1$. Furthermore, if for every $\varphi \in \text{Aut}_0(M)$ we have $a_\varphi = 0$ and there exists $\varphi_0 \in \text{Aut}_0(M)$ with $\lambda_{\varphi_0} \neq 1$, the group $\text{Aut}_0(M)$ becomes linear after applying a transformation of the form

$$\begin{aligned} z &\mapsto \frac{z}{1 + qw}, \\ w &\mapsto \frac{w}{1 + qw}, \end{aligned} \tag{1.4}$$

for some $q \in \mathbb{R}$.

To prove that $a_\varphi = 0$ for a fixed $\varphi = (f_\varphi, g_\varphi)$ in the given coordinates, we introduce weights as follows. Let each of $z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n$ be of weight 1 and u be of weight 2. Then we can write a weight decomposition for the function F as follows

$$F(z, \bar{z}, u) = \sum_{j=\gamma}^{\infty} F_j,$$

where F_j is the component of F of weight j , and $F_\gamma \neq 0$. Next, since φ is a local automorphism of M , we have

$$\text{Im } g_\varphi = \langle f_\varphi, f_\varphi \rangle + F(f_\varphi, \bar{f}_\varphi, \text{Re } g_\varphi), \tag{1.5}$$

where we set $v = \langle z, z \rangle + F(z, \bar{z}, u)$. Extracting all terms of weight $\gamma + 1$ from identity (1.5), we obtain the following identity (see [B], [L1], [L2])

$$\begin{aligned} &\text{Re} \left(i\tilde{g}_{\gamma+1} + 2\langle \lambda_\varphi^{-1} U_\varphi^{-1} \tilde{f}_\gamma, z \rangle \right) |_{v=\langle z, z \rangle} + T(F_\gamma, a_\varphi) = \\ &F_{\gamma+1}(z, \bar{z}, u) - \frac{1}{\lambda_\varphi^2} F_{\gamma+1}(\lambda_\varphi U_\varphi z, \overline{\lambda_\varphi U_\varphi z}, \lambda_\varphi^2 u). \end{aligned} \tag{1.6}$$

Here $(\sum_{j=1}^{\infty} \tilde{f}_j, \sum_{j=1}^{\infty} \tilde{g}_j)$ is the weight decomposition for the map $(\tilde{f}, \tilde{g}) := (f_{\varphi} - f_{\varphi}^Q, g_{\varphi} - g_{\varphi}^Q)$, where $\varphi^Q = (f_{\varphi}^Q, g_{\varphi}^Q)$ is the following local automorphism of the hyperquadric given by the equation $v = \langle z, z \rangle$

$$\begin{aligned} z &\mapsto \frac{\lambda_{\varphi} U_{\varphi}(z + a_{\varphi} w)}{1 - 2i\langle z, a \rangle - (r_{\varphi} + i\langle a, a \rangle)w}, \\ w &\mapsto \frac{\sigma_{\varphi} \lambda_{\varphi}^2 w}{1 - 2i\langle z, a \rangle - (r_{\varphi} + i\langle a, a \rangle)w}, \end{aligned}$$

and

$$\begin{aligned} T(F_{\gamma}, a_{\varphi}) &:= 2\operatorname{Re} \left(-2i\langle z, a_{\varphi} \rangle F_{\gamma} + (u + i\langle z, z \rangle) \sum_{j=1}^n a_j \frac{\partial F_{\gamma}}{\partial z_j} + \right. \\ &\quad \left. 2i\langle z, a \rangle \sum_{j=1}^n z_j \frac{\partial F_{\gamma}}{\partial z_j} + i\langle z, a_{\varphi} \rangle (u + i\langle z, z \rangle) \frac{\partial F_{\gamma}}{\partial u} \right), \end{aligned}$$

where a_1, \dots, a_n denote the components of the vector a_{φ} .

If $F_{\gamma+1} = 0$, the right-hand side of (1.6) vanishes, and the proof of Proposition 1 in [L1] shows that the resulting homogeneous identity can only hold if $a_{\varphi} = 0$. Clearly, if $F_{k\bar{l}} \equiv 0$ for $k \neq \bar{l}$, then the weight decomposition for F contains only terms of even weights, and, in particular, we have $F_{\gamma+1} = 0$. Thus, we have shown that $\operatorname{Aut}_0(M)$ is linearizable if $d_0(M) = n^2$.

Observe further that if $d_0(M) = n^2$, the mapping Λ defined in (1.2) is constant, that is, $\lambda_{\varphi} = 1$ for all $\varphi \in \operatorname{Aut}_0(M)$. Indeed, consider the restriction of Λ to $U(n-m, m)$. Every element $U \in U(n-m, m)$ can be represented as $U = e^{i\psi} \cdot V$ with $\psi \in \mathbb{R}$ and $V \in SU(n-m, m)$. Note that there are no non-trivial homomorphisms from the unit circle into \mathbb{R}_+ since \mathbb{R}_+ has no non-trivial compact subgroups. Also, there are no non-trivial homomorphisms from $SU(n-m, m)$ into \mathbb{R}_+ since the kernel of any such homomorphism is a proper normal subgroup of $SU(n-m, m)$ of positive dimension, and $SU(n-m, m)$ is a simple group. Thus, Λ is constant on $U(n-m, m)$ and hence on all of $G_0(M)$. It then follows that, in coordinates in which $\operatorname{Aut}_0(M)$ is linear, the function F is invariant under all linear transformations of the z -variables from $U(n-m, m)$ and therefore depends only on $\langle z, z \rangle$ and u . Conditions (1.1) imply that $F_{2\bar{2}} \equiv 0$, $F_{3\bar{3}} \equiv 0$. Thus, F has the form

$$F(z, \bar{z}, u) = \sum_{k=4}^{\infty} C_k(u) \langle z, z \rangle^k, \quad (1.7)$$

where $C_k(u)$ are real-valued analytic functions of u , and for some k we have $C_k(u) \not\equiv 0$. Note, in particular, that if $d_0(M) = n^2$, then 0 is an umbilic point of M .

Conversely, if M is given in the normal form by an equation

$$v = \langle z, z \rangle + F(z, \bar{z}, u),$$

with $F \not\equiv 0$ of the form (1.7), then $\text{Aut}_0(M)$ contains all linear transformations (1.3) with $U \in U(n-m, m)$, $\lambda = 1$ and $\sigma = 1$, and therefore $d_0(M) = n^2$. For $n > 2m$ and for $n = 2m$ with $G_0(M) = U(m, m)$, $\text{Aut}_0(M)$ clearly coincides with the group of all transformations of the form

$$\begin{aligned} z &\mapsto Uz, \\ w &\mapsto w. \end{aligned} \tag{1.8}$$

where $U \in U(n-m, m)$. If $n = 2m$ and $G_0(M) = U'(m, m)$, then $\text{Aut}_0(M)$ consists of all mappings

$$\begin{aligned} z &\mapsto Uz, \\ w &\mapsto \sigma w, \end{aligned}$$

where $U \in U'(m, m)$, $\langle Uz, Uz \rangle = \sigma \langle z, z \rangle$, $\sigma = \pm 1$.

We will now concentrate on the case $0 < d_0(M) < n^2$ (hence assuming that $n \geq 2$). For the strongly pseudoconvex case we obtain the following

THEOREM 1.1 *Let M be a strongly pseudoconvex real-analytic non-spherical hypersurface in \mathbb{C}^{n+1} with $n \geq 2$ (here $m = 0$) given in normal coordinates in which $\text{Aut}_0(M)$ is linear. Then the following holds*

- (i) $d_0(M) \geq n^2 - 2n + 3$ implies $d_0(M) = n^2$;
- (ii) if $d_0(M) = n^2 - 2n + 2$, after a linear change of the z -coordinates the equation of M takes the form

$$v = \sum_{\alpha=1}^n |z_\alpha|^2 + F(z, \bar{z}, u), \tag{1.9}$$

where F is a function of $|z_1|^2$, $\langle z, z \rangle := \sum_{\alpha=1}^n |z_\alpha|^2$ and u :

$$F(z, \bar{z}, u) = \sum_{p+q \geq 4} C_{pq}(u) |z_1|^{2p} \langle z, z \rangle^q, \tag{1.10}$$

where $C_{pq}(u)$ are real-valued analytic functions of u , and $C_{pq}(u) \not\equiv 0$ for some p, q with $p > 0$;

(iii) if a hypersurface M is given in the form described in (ii) (without assuming the linearity of $\text{Aut}_0(M)$ a priori), the group $\text{Aut}_0(M)$ coincides with the group of all mappings of the form (1.8), where $U \in U(1) \times U(n-1)$ (with $U(1) \times U(n-1)$ realized as a group of block-diagonal matrices in the standard way).

Corollary 1.2 *If M is a strongly pseudoconvex real-analytic hypersurface in \mathbb{C}^{n+1} , and the dimension of $\text{Aut}_0(M)$ is greater than or equal to $n^2 - 2n + 2$, then the origin is an umbilic point of M .*

For the case $m \geq 1$ we prove the following

THEOREM 1.3 *Let M be a Levi non-degenerate real-analytic non-spherical hypersurface in \mathbb{C}^{n+1} with $m \geq 1$. Then the following holds*

(i) $d_0(M) \geq n^2 - 2n + 4$ implies $d_0(M) = n^2$;

(ii) if $d_0(M) = n^2 - 2n + 3$, the group $\text{Aut}_0(M)$ is linearizable and in some normal coordinates in which $\text{Aut}_0(M)$ is linear, the equation of M takes the form

$$v = 2\text{Re } z_1 \bar{z}_n + 2\text{Re } z_2 \bar{z}_{n-1} + \dots + 2\text{Re } z_m \bar{z}_{n-m+1} + \sum_{\alpha=m+1}^{n-m} |z_\alpha|^2 + F(z, \bar{z}, u), \quad (1.11)$$

where F is a function of $|z_n|^2$, $\langle z, z \rangle := 2\text{Re } z_1 \bar{z}_n + 2\text{Re } z_2 \bar{z}_{n-1} + \dots + 2\text{Re } z_m \bar{z}_{n-m+1} + \sum_{\alpha=m+1}^{n-m} |z_\alpha|^2$ and u :

$$F(z, \bar{z}, u) = \sum C_{rpq} u^r |z_n|^{2p} \langle z, z \rangle^q, \quad (1.12)$$

where at least one of $C_{rpq} \in \mathbb{R}$ is non-zero, the summation is taken over $p \geq 1$, $q \geq 0$, $r \geq 0$ such that $(r + q - 1)/p = s$ with $s \geq -1/2$ being a fixed rational number, and

$$F(z, \bar{z}, u) = \sum_{k, \bar{l} \geq 2} F_{k\bar{l}}(z, \bar{z}, u),$$

where $F_{2\bar{3}} = 0$ and identities (1.1) hold for $F_{2\bar{2}}$ and $F_{3\bar{3}}$;

(iii) if a hypersurface M is given in the form described in (ii) (without assuming the linearity of $\text{Aut}_0(M)$ a priori), the group $\text{Aut}_0(M)$ coincides with the group of all mappings of the form

$$\begin{aligned} z &\mapsto |\mu|^{1/(s+1)} U z, \\ w &\mapsto |\mu|^{2/(s+1)} w, \end{aligned} \quad (1.13)$$

with $U \in S$, where S is the group introduced in Lemma 3.1 below, and μ is a parameter in this group (see formula (3.2)).

Corollary 1.4 *Let M be a Levi non-degenerate real-analytic hypersurface in \mathbb{C}^{n+1} , with $n \geq 2$ and $m \geq 1$, and assume that the dimension of $\text{Aut}_0(M)$ is greater than or equal to $n^2 - 2n + 3$. If the origin is a non-umbilic point of M , then in some normal coordinates the equation of M takes the form*

$$\begin{aligned} v = & 2\text{Re } z_1 \bar{z}_n + 2\text{Re } z_2 \bar{z}_{n-1} + \dots + \\ & 2\text{Re } z_m \bar{z}_{n-m+1} + \sum_{\alpha=m+1}^{n-m} |z_\alpha|^2 \pm |z_n|^4. \end{aligned} \quad (1.14)$$

We remark that hypersurfaces (1.14) occur in [P] in connection with studying unbounded homogeneous domains in complex space.

The proofs of Theorems 1.1 and 1.3 are given in Sections 2 and 3 respectively. Before proceeding we wish to acknowledge that this work was initiated while the first author was visiting the Mathematical Sciences Institute of the Australian National University.

2 The Strongly Pseudoconvex Case

First of all, we note that in this case the mapping Λ defined in (1.2) is constant, that is, $\lambda_\varphi = 1$ for all $\varphi \in \text{Aut}_0(M)$. This follows from the fact that all eigenvalues of U_φ are unimodular, or, alternatively, from the compactness of $G_0(M)$ and the observation that \mathbb{R}_+ does not have non-trivial compact subgroups. Next, by a linear change of the z -coordinates the matrix H can be transformed into the identity matrix E_n , and for the remainder of this section we assume that $H = E_n$. Hence we assume that the equation of M is written in the form (1.9), where the function F satisfies the normal form conditions.

It is shown in Lemma 2.1 of [IK] that any closed connected subgroup of the unitary group $U(n)$ of dimension $n^2 - 2n + 3$ or larger is either $SU(n)$ or $U(n)$ itself. Hence, if $d_0(M) \geq n^2 - 2n + 3$, we have $G_0(M) \supset SU(n)$, and therefore $F(z, \bar{z}, u)$ is invariant under all linear transformations of the z -variables from $SU(n)$. This

implies that $F(z, \bar{z}, u)$ is a function of $\langle z, z \rangle$ and u , which gives that $F(z, \bar{z}, u)$ is invariant under the action of the full unitary group $U(n)$ and thus $d_0(M) = n^2$, as stated in (i).

The proof of part (ii) of the theorem is also based on Lemma 2.1 of [IK]. For the case $d_0(M) = n^2 - 2n + 2$ the lemma gives that the connected identity component G_0^c of G_0 is either conjugate in $U(n)$ to the subgroup $U(1) \times U(n-1)$ realized as block-diagonal matrices, or, for $n = 4$, contains a subgroup conjugate in $U(n)$ to $Sp_{2,0}$. If the latter is the case, then, since $Sp_{2,0}$ acts transitively on the sphere of dimension 7 in \mathbb{C}^4 , $F(z, \bar{z}, u)$ is a function of $\langle z, z \rangle$ and u , which implies that $F(z, \bar{z}, u)$ is invariant under the action of the full unitary group $U(4)$ and thus $d_0(M) = 16$, which is impossible. Hence G_0^c is conjugate to $U(1) \times U(n-1)$, and therefore, after a unitary change of the z -coordinates, the equation of M can be written in the form (1.9) where the function F depends on $|z_1|^2$, $\langle z, z \rangle' := \sum_{\alpha=2}^n |z_\alpha|^2$ and u . Clearly, $\langle z, z \rangle' = \langle z, z \rangle - |z_1|^2$, and F can be written as a function of $|z_1|^2$, $\langle z, z \rangle$ and u as in (1.10). Next, conditions (1.1) imply that $F_{2\bar{2}} \equiv 0$, $F_{3\bar{3}} \equiv 0$, and thus the summation in (1.10) is taken over p, q such that $p+q \geq 4$. Further, if $C_{pq} \equiv 0$ for all $p > 0$, F has the form (1.7) and therefore $G_0 = U(n)$ which is impossible. Thus for some p, q with $p > 0$ we have $C_{pq} \not\equiv 0$, and (ii) is established.

If M is given in the normal form and is written as in (1.9), (1.10), $\text{Aut}_0(M)$ clearly contains all maps of the form (1.8) with $U \in U(1) \times U(n-1)$. Hence $d_0(M) \geq n^2 - 2n + 2$. If $d_0(M) > n^2 - 2n + 2$, then by part (i) of the theorem, $d_0(M) = n^2$ and hence $G_0(M) = U(n)$. Then F has the form (1.7) which is impossible because for some p, q with $p > 0$ the function C_{pq} does not vanish identically. Thus $d_0(M) = n^2 - 2n + 2$, and hence $G_0^c(M) = U(1) \times U(n-1)$. It is not hard to show that $G_0(M)$ is connected (note, for example, that by an argument given in the introduction, $\text{Aut}_0(M)$ is linear in these coordinates), and therefore $\text{Aut}_0(M)$ coincides with the group of all mappings of the form (1.8), where $U \in U(1) \times U(n-1)$.

Thus, (iii) is established, and the theorem is proved. \square

3 The Case of $m \geq 1$

We start with the following algebraic lemma.

Lemma 3.1 *Let $G \subset U(n-m, m)$ be a connected real algebraic subgroup of $GL_n(\mathbb{C})$, $n \geq 2m$, $m \geq 1$, with Hermitian form pre-*

served by $U(n-m, m)$ written as

$$\begin{pmatrix} & & & & 1 \\ & 0 & & & \\ & & & \ddots & \\ & & & 1 & \\ & & E_{n-2m} & & \\ & & 1 & & \\ & \ddots & & & 0 \\ 1 & & & & \end{pmatrix}, \quad (3.1)$$

where E_{n-2m} is the $(n-2m) \times (n-2m)$ identity matrix, and the number of 1's on each side of E_{n-2m} is m . Then the following holds

(a) if $\dim G \geq n^2 - 2n + 4$, we have either $G = SU(n-m, m)$, or $G = U(n-m, m)$;

(b) if $\dim G = n^2 - 2n + 3$, the group G either is conjugate in $U(n-m, m)$ to the group S that consists of all matrices of the form

$$\begin{pmatrix} \mu & -\mu\bar{x}^T H' A & c \\ 0 & A & x \\ 0 & 0 & 1/\bar{\mu} \end{pmatrix}, \quad (3.2)$$

where $\mu, c \in \mathbb{C}$, $\mu \neq 0$, $x \in \mathbb{C}^{n-2}$, $A \in U(n-m-1, m-1)$ (i.e., A is an $(n-2) \times (n-2)$ -matrix with complex elements such that $A^T H' \bar{A} = H'$ with H' obtained from matrix (3.1) by removing the first and the last columns and rows), and the following holds

$$2\operatorname{Re} \frac{c}{\mu} + x^T H' \bar{x} = 0,$$

or, if $n = 4$ and $m = 2$, coincides with $e^{i\mathbb{R}}(Sp_4(B, \mathbb{C}) \cap SU(2, 2))$,
or, if $n = 2$ and $m = 1$, coincides with $SU(1, 1)$. Here the subgroup $Sp_4(B, \mathbb{C}) \subset GL_4(\mathbb{C})$ consists of matrices preserving a non-degenerate skew-symmetric bilinear form B equivalent to the form given by the matrix

$$B_0 := \begin{pmatrix} 0 & E_2 \\ -E_2 & 0 \end{pmatrix}, \quad (3.3)$$

where E_2 is the 2×2 identity matrix,

Proof: Let $V \subset U(n-m, m)$ be a real algebraic subgroup of $GL_n(\mathbb{C})$ such that $\dim V \geq n^2 - 2n + 3$. Consider $V_1 := V \cap SU(n-m, m)$. Clearly, $\dim V_1 \geq n^2 - 2n + 2$. Let $V_1^{\mathbb{C}} \subset SL_n(\mathbb{C})$ be the complexification of V_1 . We have $\dim_{\mathbb{C}} V_1^{\mathbb{C}} \geq n^2 - 2n + 2$. Consider the maximal complex closed subgroup $W(V) \subset SL_n(\mathbb{C})$ that contains $V_1^{\mathbb{C}}$. Clearly, $\dim_{\mathbb{C}} W(V) \geq n^2 - 2n + 2$. All closed maximal subgroups of $SL_n(\mathbb{C})$ had been classified (see [D]), and the lower bound on the dimension of $W(V)$ gives that either $W(V) = SL_n(\mathbb{C})$, or $W(V)$ is conjugate to one of the parabolic subgroups

$$P^1 := \left\{ \begin{pmatrix} 1/\det C & b \\ 0 & C \end{pmatrix}, b \in \mathbb{C}^{n-1}, C \in GL_{n-1}(\mathbb{C}) \right\},$$

$$P^2 := \left\{ \begin{pmatrix} C & b \\ 0 & 1/\det C \end{pmatrix}, b \in \mathbb{C}^{n-1}, C \in GL_{n-1}(\mathbb{C}) \right\}$$

(note that $P^1 = P^2$ for $n = 2$), or, for $n = 4$, $W(V)$ is conjugate to $Sp_4(\mathbb{C})$.

Suppose that for some $g \in SL_n(\mathbb{C})$ and $j \in \{1, 2\}$ we have $g^{-1}W(V)g = P^j$. It is not hard to show that, due to the lower bound on the dimension of $W(V)$, g can be chosen to belong to $SU(n-m, m)$. Then $g^{-1}V_1g \subset P^j \cap SU(n-m, m)$. It is easy to compute the intersections $P^j \cap SU(n-m, m)$ for $j = 1, 2$ and see that they are equal and coincide with the group S_1 of matrices of the form (3.2) with determinant 1. Clearly, $\dim S_1 = n^2 - 2n + 2 \leq \dim V_1$ and therefore V_1 is conjugate to S_1 in $SU(n-m, m)$.

Suppose now that $n = 4$ and for some $g \in SL_4(\mathbb{C})$ we have $g^{-1}W(V)g = Sp_4(\mathbb{C})$. In particular, $g^{-1}V_1g \subset Sp_4(\mathbb{C}) \cap g^{-1}SU(4-m, m)g$ (here we have either $m = 1$, or $m = 2$). It can be shown that $\dim Sp_4(\mathbb{C}) \cap g^{-1}SU(3, 1)g \leq 6$ for all $g \in SL_4(\mathbb{C})$. At the same time we have $\dim V_1 \geq 10$. Hence $W(V)$ in fact cannot be conjugate to $Sp_4(\mathbb{C})$, if $m = 1$. Therefore, $m = 2$, and $V_1 \subset gSp_4(\mathbb{C})g^{-1} \cap SU(2, 2) = Sp_4(B, \mathbb{C}) \cap SU(2, 2)$, where B is some non-degenerate skew-symmetric bilinear form. It is straightforward to show that $Sp_4(B, \mathbb{C}) \cap SU(2, 2)$ is connected and $\dim Sp_4(B, \mathbb{C}) \cap SU(2, 2) \leq 10$. Therefore $V_1 = Sp_4(B, \mathbb{C}) \cap SU(2, 2)$.

Suppose now that $\dim G \geq n^2 - 2n + 4$. Then $\dim G_1 \geq n^2 - 2n + 3$, and the above considerations give that $W(G) = SL_n(\mathbb{C})$. Hence $G_1 = SU(n-m, m)$ which implies that either $G = SU(n-m, m)$, or $G = U(n-m, m)$, thus proving (a).

Let $\dim G = n^2 - 2n + 3$. In this case we have either $\dim G_1 = n^2 - 2n + 2$, or $G = SU(1, 1)$, if $n = 2$, $m = 1$. In the first case we obtain that G_1 either is conjugate to S_1 in $SU(n-m, m)$, or,

for $n = 4$ and $m = 2$ coincides with $Sp_4(B, \mathbb{C}) \cap SU(2, 2)$ for some non-degenerate skew-symmetric bilinear form B equivalent to the form B_0 defined in (3.3). This gives that G in the first case either is conjugate to S in $U(n - m, m)$, or for $n = 4$ and $m = 2$ coincides with $e^{i\mathbb{R}}(Sp_4(B, \mathbb{C}) \cap SU(2, 2))$, and (b) is established.

The lemma is proved. \square

We will now prove Theorem 1.3. Suppose first that $d_0(M) \geq n^2 - 2n + 4$ and assume that H is written in the diagonal form with 1's in the first $n - m$ positions and -1 's in the last m positions on the diagonal. Lemma 3.1 gives that either $G_0^c(M) = SU(n - m, m)$ (in which case $n \geq 3$), or $G_0(M) = U(n - m, m)$, or, for $n = 2m$, $G_0(M) = U'(m, m)$. If $G_0(M) \supset U(n - m, m)$, then $d_0(M) = n^2$, and (i) is established. Assume that $G_0(M) \supset SU(n - m, m)$. Suppose that $m \geq 2$. Then G_0 contains the product $R := SU(n - m) \times SU(m)$ realized as block-diagonal matrices. Arguing as in the introduction, we obtain that in some normal coordinates all elements of the compact group $\hat{R} := \Phi^{-1}(R)$ can be written in the form (1.8) and thus F is a function of $\langle z, z \rangle_+ := \sum_{j=1}^{n-m} |z_j|^2$, $\langle z, z \rangle_- := \sum_{j=n-m+1}^n |z_j|^2$, and u . Hence all elements of odd weight in the weight decomposition for F are zero. This shows that $F_{\gamma+1} \equiv 0$, and identity (1.6) again implies that $\text{Aut}_0(M)$ becomes linear after a change of coordinates of the form (1.4). If $m = 1$, $\text{Aut}_0(M)$ is linearizable by [Ezh1], [Ezh2].

Therefore, there exist normal coordinates where the corresponding function F is invariant under all linear transformations of the z -variables from $SU(n - m, m)$. This implies that F is in fact invariant under all linear transformations of the z -variables from $U(n - m, m)$. Hence $d_0(M) = n^2$, and (i) is established.

Suppose now that $d_0(M) = n^2 - 2n + 3$. By a linear change of the z -coordinates the matrix H can be transformed into matrix (3.1), and from now on we assume that H is given in this form. Hence the equation of M is written as in (1.11), where the function F satisfies the normal form conditions. Arguing as in the preceding paragraph, we see that for $n = 2$, $m = 1$, the group G_0^c cannot coincide with $SU(1, 1)$. Assume first that after a linear change of the z -coordinates preserving the form H the group $G_0^c(M)$ coincides with S . Then $G_0(M)$ contains the compact subgroup $Q = \{e^{it} \cdot E_n, t \in \mathbb{R}\}$, where E_n is the $n \times n$ identity matrix. The argument based on identity (1.6) that we gave in the introduction, again yields that $\text{Aut}_0(M)$ is linearizable. Passing to coordinates in which $\text{Aut}_0(M)$ is linear, we obtain that for every $U \in S$ the equation of

M is invariant under the linear transformation

$$\begin{aligned} z &\mapsto \lambda_U U z, \\ w &\mapsto \lambda_U^2 w, \end{aligned} \tag{3.4}$$

where $\lambda_U = \Lambda(U)$. The group S contains $U(n - m - 1, m - 1)$ realized as the subgroup of all matrices of the form (3.2) with $\mu = 1$, $c = 0$, $x = 0$. Since Λ is constant on $U(n - m - 1, m - 1)$, we have $\lambda_U = 1$ for all $U \in U(n - m - 1, m - 1)$. Therefore, the function $F(z, \bar{z}, u)$ depends on $z_1, z_n, \bar{z}_1, \bar{z}_n, \langle z, z \rangle' := 2\operatorname{Re} z_2 \bar{z}_{n-1} + \dots + 2\operatorname{Re} z_m \bar{z}_{n-m+1} + \sum_{\alpha=m+1}^{n-m} |z_\alpha|^2$ and u . Clearly, $\langle z, z \rangle' = \langle z, z \rangle - 2\operatorname{Re} z_1 \bar{z}_n$, and F can be written as follows

$$F(z, \bar{z}, u) = \sum_{r, q \geq 0} D_{rq}(z_1, z_n, \bar{z}_1, \bar{z}_n) u^r \langle z, z \rangle^q,$$

where D_{rq} are real-analytic.

We will now determine the form of the functions D_{rq} . The group S contains the subgroup I of all matrices as in (3.2) with $|\mu| = 1$, $x = 0$ and $A = E_{n-2}$, where E_{n-2} is the $(n - 2) \times (n - 2)$ identity matrix. Since every eigenvalue of any $U \in I$ is unimodular, we have $\lambda_U = 1$ for all $U \in I$, and therefore D_{rq} is invariant under all linear transformations from I . It is straightforward to show (see also [Ezh2]) that any polynomial of $z_1, z_n, \bar{z}_1, \bar{z}_n$ invariant under all linear transformations from I is a function of $\operatorname{Re} z_1 \bar{z}_n$ and $|z_n|^2$, and hence every D_{rq} has this property. Let further J be the subgroup of S given by the conditions $\mu = 1$, $A = E_{n-2}$. For every $U \in J$ we also have $\lambda_U = 1$, and hence D_{rq} is invariant under all linear transformations from J . It is then easy to see that D_{rq} has to be a function of $|z_n|^2$ alone. Thus, the function F has the form (1.12), and it remains to show that the summation in (1.12) is taken over $p \geq 1, q \geq 0, r \geq 0$ such that $(r + q - 1)/p = s$, where $s \geq -1/2$ is a fixed rational number.

Let K be the 1-dimensional subgroup of S given by the conditions $\mu > 0, c = 0, x = 0, A = E_{n-2}$. It is straightforward to show that every homomorphism $\Psi : K \rightarrow \mathbb{R}_+$ has the form $U \mapsto \mu^\alpha$, where $\alpha \in \mathbb{R}$. Considering $\Psi = \Lambda|_K$ we obtain that there exists $\alpha \in \mathbb{R}$ such that for every $U \in K$ we have $\lambda_U = \mu^\alpha$. We will now prove that $\alpha \neq 0$. Indeed, otherwise F would be invariant under all linear transformations from K and therefore would be a function of $\langle z, z \rangle$ and u , which implies that $G_0(M) \supset U(n - m, m)$. This contradiction shows that $\alpha \neq 0$ and hence $\lambda_U \neq 1$ for every $U \in K$ with $\mu \neq 1$.

Plugging a mapping of the form (3.4) with $U \in K$, $\mu \neq 1$, into equation (1.11), where $F \neq 0$ has the form (1.12) we obtain that, if $C_{rpq} \neq 0$, then

$$\lambda_U^{r+p+q-1} = \mu^p. \quad (3.5)$$

The equation of M is written in the normal form, hence $p + q \geq 2$ and $r + p + q - 1 \geq 1$. Since $\lambda_U \neq 1$, we obtain that $p \geq 1$. Further, (3.5) implies

$$\lambda_U^{(r+p+q-1)/p} = \mu,$$

and, since the right-hand side in the above identity does not depend on r, p, q , for all non-zero coefficients C_{rpq} the ratio $(r + q - 1)/p$ must have the same value; we denote it by s . Clearly, s is a rational number and $s \geq -1/2$. We also remark that $\alpha = p/(r + p + q - 1) = 1/(s + 1)$.

Assume now that $n = 4$, $m = 2$ and $G_0^c(M)$ coincides with $e^{i\mathbb{R}}(Sp_4(B, \mathbb{C}) \cap SU(2, 2))$ for some non-degenerate skew-symmetric non-degenerate bilinear form B equivalent to the form B_0 defined in (3.3). Then $G_0(M)$ contains the compact subgroup $Q = \{e^{it} \cdot E_4, t \in \mathbb{R}\}$, where E_4 is the 4×4 identity matrix. Arguing as above, we obtain that $\text{Aut}_0(M)$ is linearizable. Further, it is straightforward to prove that $Sp_4(B, \mathbb{C}) \cap SU(2, 2)$ is a real form of $Sp_4(B, \mathbb{C})$ and therefore is simple. Hence there does not exist a non-trivial homomorphism from $Sp_4(B, \mathbb{C}) \cap SU(2, 2)$ into \mathbb{R}_+ . Further, since \mathbb{R}_+ does not have non-trivial compact subgroups, any homomorphism from the unit circle into \mathbb{R}_+ is constant. Hence Λ is constant on $G_0(M)$. This implies that F is invariant under all linear transformations from $Sp_4(B, \mathbb{C}) \cap SU(2, 2)$. It can be shown that this group acts transitively on any pseudosphere in \mathbb{C}^4 given by the equation $\langle z, z \rangle = r$, which yields that F is a function of $\langle z, z \rangle$ and u and hence $d_0(M) = n^2$. This contradiction proves that in fact $G_0^c(M) \neq e^{i\mathbb{R}}(Sp_4(B, \mathbb{C}) \cap SU(2, 2))$ for $n = 4$, $m = 2$. Thus, (ii) is established.

Suppose that M is given in the normal form, written as in (1.11), (1.12), and the summation in (1.12) is taken over $p \geq 1$, $q \geq 0$, $r \geq 0$ such that $(r + q - 1)/p = s$, where $s \geq -1/2$ is a fixed rational number. Set $\alpha = 1/(s + 1)$ and for every $U \in S$ define $\lambda_U = |\mu|^\alpha$. It is then straightforward to verify that every mapping of the form (3.4) with $U \in S$ is an automorphism of M . Therefore, $G_0(M)$ contains S and hence $d_0(M) \geq n^2 - 2n + 3$. If $d_0(M) > n^2 - 2n + 3$, then by part (i) of the theorem, $d_0(M) = n^2$ and hence $G_0(M) \supset U(n - m, m)$. Then F is a function of $\langle z, z \rangle$ and u , which is impossible since for every non-zero C_{rpq} we have $p \geq 1$. Hence $d_0(M) = n^2 - 2n + 3$ and hence $G_0^c(M) = S$. Finally, observe that

by an argument given in the introduction, $\text{Aut}_0(M)$ is linear in these coordinates. It is now straightforward to show that $\text{Aut}_0(M)$ coincides with the group of all mappings of the form (1.13).

Thus, (iii) is established, and the theorem is proved. \square

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School of Mathematics and Statistics
 University of South Australia
 Mawson Lakes Blvd
 Mawson Lakes
 South Australia 5091
 AUSTRALIA
 E-mail: vladimir.ejov@unisa.edu.au

Department of Mathematics
 The Australian National University
 Canberra, ACT 0200
 AUSTRALIA
 E-mail: alexander.isaev@maths.anu.edu.au